

Critical Effects in Population Dynamics of Trapped Bose-Einstein Condensates

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Abstract

The population dynamics of a trapped Bose-Einstein condensate, subject to the action of an external field, is studied. This field produces a spatio-temporal modulation of the trapping potential with the frequency close to the transition frequency between the ground state and a higher energy level. For the evolution equations of fractional populations, a critical line is found. It is demonstrated that there exists a direct analogy between dynamical instability at this line and critical phenomena at a critical line of an averaged system. The related critical indices are calculated. The spatio-temporal evolution of atomic density is analyzed.

1 Introduction

A dilute cloud of Bose-condensed atoms confined within a trapping potential at low temperatures can be described by a wave function satisfying the Gross-Pitaevskii equation [1,2]. This equation is nonlinear due to the atomic interactions through a local delta potential. The mathematical structure of the equation is that of the nonlinear Schrödinger equation. Stationary states of the latter, because of the confinement caused by a trapping potential, are restricted to discrete energy levels. These stationary eigenstates form a set of *nonlinear modes*, in analogy to linear modes that are solutions of a linear Schrödinger equation. The nonlinear modes can be called *coherent modes*, since the wave function of the Gross-Pitaevskii equation corresponds to a coherent state of Bose-condensed atoms. It is also possible to use the term *topological modes*, emphasizing that the wave functions related to different energy levels have different spatial topology.

This modal structure of the confined condensate states is in close analogy with *nonlinear optical modes* of nonlinear waveguide equations in optics [3,4]. And the methods of creating nonlinear coherent modes of trapped Bose atoms [5-7] are also similar to those employed in optics, where one uses specially prepared initial conditions or invokes an action of external fields. Dynamics of fractional populations, characterizing the occupation of coherent modes of Bose-condensed trapped atoms, displays as well many phenomena equivalent to those known in optics, for instance collapses, revivals, and Rabi-type oscillations. The nonlinear dynamics of processes coupling bosonic coherent modes have been studied in several papers considering the general case [5-7], antisymmetric modes [8,9], dipole topological modes in a two-component condensate [10], dark-soliton states [11], and vortex modes [12-19].

The aim of the present paper is threefold: First, we demonstrate the existence of a critical line for the population dynamics of Bose condensates, which is a kind of bifurcation line due to the nonlinearity of evolution equations. Second, we show that there is an intimate relation between the studied nonlinear dynamical system and the corresponding averaged system, so that the bifurcation line is an analog of the critical line, and instabilities happening at the former are the counterparts of critical phenomena occurring at the latter. Third, we study the spatio-temporal evolution of atomic density under the action of a resonant external field.

2 Critical Phenomena

The time-dependent Gross-Pitaevskii equation has the form

$$i\hbar \frac{\partial \varphi}{\partial t} = [\hat{H}(\varphi) + \hat{V}] \varphi, \quad \hat{H}(\varphi) = -\frac{\hbar^2}{2m_0} \nabla^2 + U(\mathbf{r}) + AN|\varphi|^2, \quad (1)$$

where $U(\mathbf{r})$ is a trapping potential; $A \equiv 4\pi\hbar^2 a_s/m_0$; a_s is a scattering length; m_0 is mass; and N is the number of particles; \hat{V} is a potential of external fields. The wave function φ is normalized to unity, $||\varphi|| = 1$. The nonlinear topological modes are defined [5-7] as the eigenfunctions of the nonlinear Hamiltonian, that is, they are the solutions to the eigenproblem

$$\hat{H}(\varphi_n)\varphi_n(\mathbf{r}) = E_n\varphi_n(\mathbf{r}).$$

It is worth stressing here the principal difference between the topological coherent modes and collective excitations. The former are self-consistent *atomic* states defined by the *nonlinear* Gross-Pitaevskii equation. While the latter are the *elementary* excitations corresponding to small deviations from a given atomic state and are described by the *linear* Bogolubov-De Gennes equations [1,2].

Assume that at the initial time the system was condensed to the ground-state level with an energy E_0 . So that the initial condition to Eq. (1) is $\varphi(\mathbf{r}, 0) = \varphi_0(\mathbf{r})$. Suppose we wish to couple the ground state φ_0 with another state φ_j having a higher energy E_j . The best way for doing this is, clearly, by switching on an external field, say $\hat{V} = V(\mathbf{r}) \cos \omega t$, oscillating with a frequency ω which is close to the transition frequency $\omega_j \equiv (E_j - E_0)/\hbar$, so that the detuning $\Delta\omega \equiv \omega - \omega_j$ be small, $|\Delta\omega/\omega| \ll 1$. Then one can look for a solution to Eq. (1) in the form

$$\varphi(\mathbf{r}, t) = c_0(t)\varphi_0(\mathbf{r}) e^{-iE_0 t/\hbar} + c_j(t)\varphi_j(\mathbf{r}) e^{-iE_j t/\hbar} . \quad (2)$$

The considered situation is very similar to the description of nonlinear resonant processes in optics [3,4]. The validity of the quasi-resonant two-level approximation (2) for the time-dependent Gross-Pitaevskii equation has been confirmed mathematically [5-7] and also proved by direct numerical simulations of the Gross-Pitaevskii equation [8,10,12-14], the agreement between the two-level picture and the simulations being excellent.

The coefficients $c_i(t)$ define the fractional level populations $n_i(t) \equiv |c_i(t)|^2$. The equations for these coefficients can be obtained by substituting the presentation (2) in Eq. (1). It is again worth noticing the similarity of this presentation with the slowly-varying-amplitude approximation commonly used in optics, if one treats the factors $c_i(t)$ as slow functions of time, as compared to the exponentials in Eq. (2), which implies that $|dc_i/dt| \ll E_i$. This approximation, being complimented by the averaging technique [20], permits one to slightly simplify the evolution equations for $c_i(t)$. In this way [5-7], we come to the equations

$$\frac{dc_0}{dt} = -i\alpha n_j c_0 - \frac{i}{2} \beta c_j e^{i\Delta\omega t} , \quad \frac{dc_j}{dt} = -i\alpha n_0 c_j - \frac{i}{2} \beta^* c_0 e^{-i\Delta\omega t} , \quad (3)$$

in which the transition amplitudes

$$\alpha_{ij} \equiv A \frac{N}{\hbar} \int |\varphi_i(\mathbf{r})|^2 (2|\varphi_j(\mathbf{r})|^2 - |\varphi_i(\mathbf{r})|^2) d\mathbf{r} , \quad \beta \equiv \frac{1}{\hbar} \int \varphi_0^*(\mathbf{r}) V(\mathbf{r}) \varphi_j(\mathbf{r}) d\mathbf{r} ,$$

caused by the nonlinearity and by the modulating field, respectively, are introduced, and the abbreviated notation $\alpha \equiv \alpha_{0j}$, with setting $\alpha_{0j} = \alpha_{j0}$, is used. Note that the transition amplitude β is nonzero only if the potential $V(\mathbf{r})$ depends on space variables. The initial conditions to Eqs. (3) are $c_0(0) = 1$ and $c_j(0) = 0$.

Usually, one solves such evolution equations with fixed parameters α , β and $\Delta\omega$, keeping in mind a particular realization. Instead of this, we have studied the behavior of solutions to Eqs. (3) in a wide range of varying parameters. It turned out that this behaviour is surprisingly rich exhibiting new interesting effects.

First of all, it is easy to notice that the number of free parameters in Eqs. (3) can be reduced to two by the appropriate scaling. For this purpose, we measure time in units of

α^{-1} and introduce the dimensionless parameters

$$b \equiv \frac{|\beta|}{\alpha}, \quad \delta \equiv \frac{\Delta\omega}{\alpha}.$$

It is also evident that Eqs. (3) are invariant with respect to the inversion $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$, $\Delta\omega \rightarrow -\Delta\omega$, and $t \rightarrow -t$. Therefore it is possible to fix the sign of one of the parameters, say $\alpha > 0$, since the opposite case can be obtained by the inversion. For concreteness, we shall also keep in mind that β is positive. The dimensionless detuning is assumed to always be small, $|\delta| \ll 1$. And the dimensionless transition amplitude b is varied in the region $0 \leq b \leq 1$. We have accomplished a careful analysis by numerically solving Eqs. (3). When parameter b is small, the fractional populations oscillate reminding the Rabi oscillations in optics, where $|\beta|$ would play the role of the Rabi frequency. The amplitude of these oscillations increases with increasing b . It would be more correct to say that in our case there exist *nonlinear Rabi oscillations*, as far as Eqs. (3) differ from the corresponding equations for optical two-level systems by the presence of the nonlinearity due to interatomic interactions. This nonlinearity not only slightly modifies the Rabi-type oscillations of the fractional populations but, for a particular relation between parameters, can lead to dramatic effects. By accurately analyzing the behaviour of solutions to Eqs. (3), with gradually varying parameters, we have found out that there exists the *critical line*

$$b + \delta \simeq 0.5,$$

at which the system dynamics experiences sharp changes. This is illustrated in Figs. 1 to 4, where the parameter $b = 0.4999$ is kept fixed and the critical line is crossed by varying the detuning δ . In Fig. 1, the detuning is zero, and the fractional populations display the Rabi-type oscillations. By slightly shifting the detuning to $\delta = 0.0001$ drastically changes the picture to that in Fig. 2, where the top of $n_j(t)$ and the bottom of $n_0(t)$ become flat, and the oscillation period is approximately doubled. A tiny further variation of the detuning to $\delta = 0.0001001$ yields again drastic changes to Fig. 3, where there appear the upward cusps of $n_j(t)$ and the downward cusps of $n_0(t)$. The following small increase of the detuning to $\delta = 0.00011$ squeezes the oscillation period twice, as is shown in Fig. 4. After this, making δ larger does not result in essential qualitative changes of the population behaviour. All dramatic changes in dynamics occur in a tiny vicinity of the critical line. The same phenomena happen when crossing the line $b + \delta \simeq 0.5$ at other values of parameters or if δ is fixed but b is varied. This is demonstrated in Fig.5 for varying δ and another choice of $b = 0.3$.

The unusual behaviour of the fractional populations is due to the nonlinearity of the evolution equations (3). Systems of nonlinear differential equations, as is known, can possess qualitatively different solutions for parameters differing by infinitesimally small values. The transfer from one type of solutions to another type, in the theory of dynamical systems, is, generally, termed bifurcation. At a bifurcation line, dynamical system is structurally unstable.

The second aim of our paper is to show that the found instability in the considered dynamical system is analogous to a *phase transition* in a statistical system. To elucidate this analogy for the present case, we have to consider the time-averaged features of the dynamical system given by Eqs. (3). To this end, we need, first, to define an effective

Hamiltonian generating the evolution equations. This can be done by transforming these equations to the Hamiltonian form

$$i \frac{dc_0}{dt} = \frac{\partial H_{eff}}{\partial c_0^*}, \quad i \frac{dc_j}{dt} = \frac{\partial H_{eff}}{\partial c_j^*},$$

with the effective Hamiltonian

$$H_{eff} = \alpha n_0 n_j + \frac{1}{2} \left(\beta e^{i\Delta\omega t} c_0^* c_j + \beta^* e^{-i\Delta\omega t} c_j^* c_0 \right). \quad (4)$$

An effective energy of the system can be defined as a time average of the effective Hamiltonian (4). For this purpose, Eqs. (3) can be treated by means of the averaging technique [20], as it is described in detail in Ref. [5], which provides the guiding-center solutions. Substituting the latter in Eq. (4), together with the time-averaged fractional populations, results in the effective energy

$$E_{eff} = \frac{\alpha b^2}{2\varepsilon^2} \left(\frac{b^2}{2\varepsilon^2} + \delta \right),$$

where ε is a dimensionless average frequency defined by the equation

$$\varepsilon^4(\varepsilon^2 - b^2) = (\varepsilon^2 - b^2 - \varepsilon^2 \delta)^2.$$

The effective energy, being the time average of the effective Hamiltonian (4), characterizes the average features of the system. As an *order parameter* for this averaged system, one can take the difference of the time-averaged populations,

$$\eta \equiv \bar{n}_0 - \bar{n}_j = 1 - \frac{b^2}{\varepsilon^2}.$$

The capacity of the system to store the energy pumped in by the resonant field can be described by the *pumping capacity* $C_\beta = \partial E_{eff} / \partial |\beta|$. The influence of the detuning on the order parameter is characterized by the *detuning susceptibility* $\chi_\delta = |\partial \eta / \partial \delta|$. Analyzing the behaviour of the introduced characteristics as functions of the parameters b and δ , we found out that they exhibit critical phenomena at the *critical line* $b + \delta = 0.5$, which coincides with the bifurcation line for the dynamical system. Expanding these characteristics over the small relative deviation $\tau \equiv |b - b_c|/b_c$ from the *critical point* $b_c = 0.5 - \delta$, we obtain

$$\eta - \eta_c \simeq \frac{\sqrt{2}}{2} (1 - 2\delta) \tau^{1/2}, \quad C_\beta \simeq \frac{\sqrt{2}}{8} \tau^{-1/2}, \quad \chi_\delta \simeq \frac{1}{\sqrt{2}} \tau^{-1/2}, \quad (5)$$

where $\eta_c \equiv \eta(b_c)$ and $\tau \rightarrow 0$. As is seen, the pumping capacity and detuning susceptibility display divergence at the critical point. The related *critical indices* for C_β , η , and χ_δ are equal to $1/2$. These indices satisfy the known scaling relation:

$$ind(C_\beta) + 2 ind(\eta) + ind(\chi_\delta) = 2,$$

where *ind* is the evident abbreviation for index.

In order to clarify what is the origin of the found critical effects for the studied dynamical system, let us return back to Eqs. (3). Again we pass to dimensionless notation measuring time in units of α^{-1} . By means of the substitution

$$c_0 = \left(\frac{1-p}{2}\right)^{1/2} \exp\left\{i\left(q_0 + \frac{\delta}{2}t\right)\right\}, \quad c_j = \left(\frac{1+p}{2}\right)^{1/2} \exp\left\{i\left(q_1 - \frac{\delta}{2}t\right)\right\}, \quad (6)$$

where p , q_0 , and q_1 are real functions of t , equations (3) can be reduced to the form

$$\frac{dp}{dt} = -b\sqrt{1-p^2}\sin q, \quad \frac{dq}{dt} = p + \frac{bp}{\sqrt{1-p^2}}\cos q + \delta, \quad (7)$$

in which $q \equiv q_1 - q_0$. Note that this reduction to an autonomous dynamical system is valid for arbitrary detuning δ . Moreover, this system possesses the integral of motion

$$I(p, q) = \frac{1}{2}p^2 - b\sqrt{1-p^2}\cos q + \delta p, \quad (8)$$

which can be defined by using the initial conditions $p(0) = -1$ and $q(0) = 0$ corresponding to the conditions $c_0(0) = 1$ and $c_j(0) = 0$. This gives

$$I(-1, 0) = \frac{1}{2} - \delta. \quad (9)$$

The existence of the integral of motion means that the dynamical system is integrable in quadratures. This fact does not help much for studying the time evolution of the system, since the formal solutions $p(t)$ and $q(t)$ are expressed through rather complicated integrals, so that the system evolution, anyway, is to be analysed numerically. However, the property of integrability implies that the appearance of chaos in the system is impossible. Consequently, the observed critical effects in no way could be related to chaos. Then what is their origin? The answer to this question comes from the analysis of the phase portrait for Eqs. (7) in the rectangle defined by the inequalities $-1 \leq p \leq 1$, $0 \leq q \leq 2\pi$. This analysis shows that, if $b + \delta < 0.5$, then the motion starting at the initial point $p(0) = -1$, $q(0) = 0$ is oscillatory, with a trajectory lying always in the lower part of the phase rectangle, below the separatrix given by the equation

$$\frac{1}{2}p^2 - b\sqrt{1-p^2}\cos q + \delta p - b = 0. \quad (10)$$

When $b + \delta = 0.5$, the separatrix touches the initial point, so that the following motion occurs in the phase region above the separatrix. In this way, if we consider, under the given initial conditions, the parametric manifold formed by the parameters $b \in [0, 1]$ and $\delta \ll 1$, then the critical line $b + \delta = 0.5$ separates the parametric regions related to two different types of solutions to Eqs. (7).

3 Spatio-Temporal Evolution

The resonance formation of coherent topological modes can be noticed by observing the spatio-temporal behaviour of the atomic density. To illustrate this, we consider a cylindrical trap, with the frequency ratio

$$\nu \equiv \frac{\omega_z}{\omega_r} \quad (\omega_r \equiv \omega_x = \omega_y). \quad (11)$$

Introduce dimensionless cylindrical variables

$$r \equiv \frac{\sqrt{r_x^2 + r_y^2}}{l_r}, \quad z \equiv \frac{r_z}{l_r} \quad \left(l_r \equiv \sqrt{\frac{\hbar}{m_0 \omega_r}} \right) \quad (12)$$

and the dimensionless wave function

$$\psi_{nmk}(r, \varphi, z) \equiv l_r^{3/2} \varphi_{nmk}(\mathbf{r}), \quad (13)$$

in which $n = 0, 1, 2, \dots$ is the radial quantum number; $m = 0, \pm 1, \pm 2, \dots$ is the azimuthal quantum number; and $k = 0, 1, 2, \dots$ is the axial quantum number. The effective coupling parameter in dimensionless units is

$$g \equiv \frac{AN}{\hbar \omega_r l_r^3} = 4\pi \frac{a_s}{l_r} N. \quad (14)$$

The density of particles

$$\rho(\mathbf{r}, t) \equiv N |\varphi(\mathbf{r}, t)|^2, \quad (15)$$

according to the form (2), contains both slow functions of time, $c_0(t)$ and $c_j(t)$, as well as fastly oscillating exponentials. To exclude the fastly oscillating terms, we introduce the *envelope density*

$$\bar{\rho}(\mathbf{r}, t) \equiv \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \rho(\mathbf{r}, t) dt, \quad (16)$$

where the integration is over time explicitly entering the exponentials, while c_0 and c_j are kept fixed. Defining the dimensionless envelope density

$$\rho(r, \varphi, z, t) \equiv \frac{l_r^3}{N} \bar{\rho}(\mathbf{r}, t), \quad (17)$$

we obtain

$$\rho(r, \varphi, z, t) = n_0(t) |\psi_0(r, \varphi, z)|^2 + n_j(t) |\psi_j(r, \varphi, z)|^2, \quad (18)$$

with j denoting the set $\{n, m, k\}$ of quantum numbers.

Numerical calculations for the envelope density (18) are illustrated in Figs. 6 to 11, where it is clearly seen how strongly the spatio-temporal behaviour of the density (18) depends on the values of the parameters b and δ in the vicinity of the critical line $b + \delta \simeq 0.5$. The behaviour of the density for the *radial dipole mode* ($n = 1, m = 0, k = 0$) is shown in Figs. 6 and 7. The corresponding transition frequency in units of ω_r , for large $g \gg 1$, is

$$\omega_{100} \simeq 0.096(\nu g)^{2/5}.$$

The density for the *basic vortex mode* ($n = 0, m = 1, k = 0$) is given in Figs. 8 and 9. The related transition frequency, for $g \gg 1$, is

$$\omega_{010} \simeq \frac{3.424}{(\nu g)^{2/5}}.$$

And the density (18) for the *axial dipole mode* ($n = 0$, $m = 0$, $k = 1$) is shown in Figs. 10 and 11; the transition frequency of the latter mode being

$$\omega_{001} \simeq 0.060(\nu g)^{2/5} .$$

In all Figs. 6 to 11, we set the coupling parameter (14) $g=100$ and take the frequency ratio (11) $\nu = 10$, which corresponds to a disk-shape trap with the aspect ratio

$$R_r \equiv \left(\frac{\langle r^2 \rangle_{nmk}}{\langle z^2 \rangle_{nmk}} \right)^{1/2} \simeq \sqrt{2} \nu , \quad (19)$$

as $g \gg 1$. For the chosen values of g and ν the transition frequencies are

$$\omega_{010} \simeq 0.216 \quad (\text{basic vortex mode}) ,$$

$$\omega_{001} \simeq 0.951 \quad (\text{axial dipole mode}) ,$$

$$\omega_{100} \simeq 1.521 \quad (\text{radial dipole mode}) .$$

This demonstrates that the transition frequencies are arranged in the order

$$\omega_{010} < \omega_{001} < \omega_{100} , \quad (20)$$

the lowest energy level corresponding to the basic vortex mode.

Let us note that the basic vortex mode, with the winding number $|m| = 1$, sets off from other vortices with higher winding numbers $|m| \geq 2$. For a given angular momentum, the creation of several basic vortices is more energetically profitable than the formation of one vortex with a higher winding number [2,16,18].

4 Density of Current

When a coherent topological mode is excited in a trap, the density of current, as a function of space and time, also displays a peculiar behaviour. The density of current

$$\mathbf{j}(\mathbf{r}, t) \equiv - \frac{i\hbar}{2m_0} [\varphi^*(\mathbf{r}, t) \nabla \varphi(\mathbf{r}, t) - \varphi(\mathbf{r}, t) \nabla \varphi^*(\mathbf{r}, t)] \quad (21)$$

can be rewritten as

$$\mathbf{j}(\mathbf{r}, t) = \frac{\hbar}{m_0} \text{Im} \varphi^*(\mathbf{r}, t) \nabla \varphi(\mathbf{r}, t) . \quad (22)$$

The wave function (2) is the sum

$$\varphi(\mathbf{r}, t) = \varphi_0(\mathbf{r}, t) + \varphi_j(\mathbf{r}, t) \quad (23)$$

of the terms

$$\varphi_i(\mathbf{r}, t) \equiv c_i(t) \varphi_i(\mathbf{r}) e^{-iE_i t/\hbar} , \quad (24)$$

where the index $i = 0, j$.

With the wave function (23), the density of current (21) becomes

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_0(\mathbf{r}, t) + \mathbf{j}_{top}(\mathbf{r}, t) + \mathbf{j}_{int}(\mathbf{r}, t) , \quad (25)$$

where the first term is the *ground-state current density*

$$\mathbf{j}_0(\mathbf{r}, t) \equiv \frac{\hbar}{m_0} \text{Im} \varphi_0^*(\mathbf{r}, t) \nabla \varphi_0(\mathbf{r}, t) , \quad (26)$$

the second term is the *topological current density*

$$\mathbf{j}_{top}(\mathbf{r}, t) \equiv \frac{\hbar}{m_0} \text{Im} \varphi_j^*(\mathbf{r}, t) \nabla \varphi_j(\mathbf{r}, t) , \quad (27)$$

due to the excited topological mode, and the third term is the *interference current density*

$$\mathbf{j}_{int}(\mathbf{r}, t) \equiv \frac{\hbar}{m_0} \text{Im} \left[\varphi_0^*(\mathbf{r}, t) \nabla \varphi_j(\mathbf{r}, t) + \varphi_j^*(\mathbf{r}, t) \nabla \varphi_0(\mathbf{r}, t) \right] , \quad (28)$$

caused by the interference between the ground-state mode and the excited topological mode.

For the functions (24), one has

$$\varphi_i^*(\mathbf{r}, t) \nabla \varphi_i(\mathbf{r}, t) = |c_i(t)|^2 \varphi_i^*(\mathbf{r}) \nabla \varphi_i(\mathbf{r}) .$$

Using the substitution (6), one gets

$$n_0(t) \equiv |c_0(t)|^2 = \frac{1-p(t)}{2} , \quad n_j(t) \equiv |c_j(t)|^2 = \frac{1+p(t)}{2} . \quad (29)$$

Since the ground-state wave function $\varphi_0(\mathbf{r})$ is real,

$$\mathbf{j}_0(\mathbf{r}, t) = 0 . \quad (30)$$

And the topological current density (27) acquires the form

$$\mathbf{j}_{top}(\mathbf{r}, t) = \frac{\hbar}{m_0} n_j(t) \text{Im} \varphi_j^*(\mathbf{r}) \nabla \varphi_j(\mathbf{r}) . \quad (31)$$

For example, in the case of a vortex mode, when $\varphi_j(\mathbf{r}) \sim e^{im\varphi}$, with $m = 0, \pm 1, \pm 2, \dots$, one has

$$\text{Im} \varphi_j^*(\mathbf{r}) \nabla \varphi_j(\mathbf{r}) = \frac{m}{r} |\varphi_j(\mathbf{r})|^2 \mathbf{e}_\varphi .$$

Then the topological current density (31) is

$$\mathbf{j}_{top}(\mathbf{r}, t) = \frac{m\hbar}{m_0 r} n_j(t) |\varphi_j(\mathbf{r})|^2 \mathbf{e}_\varphi . \quad (32)$$

The total topological current is, of course, zero,

$$\int \mathbf{j}_{top}(\mathbf{r}, t) d\mathbf{r} = 0 , \quad (33)$$

provided there is no flux through the boundary.

The interference current density (28) is a kind of the Josephson current oscillating with the transition frequency $\omega_j \approx \omega$. If one averages over these fast oscillations, treating $c_i(t)$ as slow functions of time, one gets

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathbf{j}_{int}(\mathbf{r}, t) dt = 0 . \quad (34)$$

Thus, the slow time variation of the summary density of current (25) is

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathbf{j}(\mathbf{r}, t) dt = \mathbf{j}_{top}(\mathbf{r}, t) , \quad (35)$$

being due to the topological current density (31).

5 Discussion

Here we have studied the resonant excitation of coherent topological modes in a trapped Bose-Einstein condensate. It is worth mentioning that there could be another way of creating those stationary modes that do not have zeros, except, may be, at infinity. Suppose, we wish to create a mode $f(\mathbf{r})$ having no zeros. Then it is possible to perturb the system with the potential

$$V_f(\mathbf{r}) = E_f - \frac{\hat{H}(f)f(\mathbf{r})}{f(\mathbf{r})} ,$$

such that the given function $f(\mathbf{r})$ be the ground-state wave function for the eigenproblem

$$\left[\hat{H}(f) + V_f(\mathbf{r}) \right] f(\mathbf{r}) = E_f f(\mathbf{r}) .$$

However, this way does not allow us to form topologically different modes having different number of zeros, which is admissible by means of the resonant excitation.

In conclusion, we have considered the population dynamics of a trapped Bose-Einstein condensate, subject to the action of a resonant spatio-temporal modulation of the trapping potential. A careful analysis of evolution equations has been made for the wide range of varying parameters. Such a variation of parameters can be easily realized by changing trap characteristics, varying the number and kind of condensed atoms, and by changing the scattering length using Feshbach resonances [1,2]. It turned out that on the manifold of possible parameters there exists a critical line where the evolution equations display structural instability. We have demonstrated that this instability for a dynamical system is analogous to a phase transition for a stationary averaged system. For the latter, one can define an order parameter, pumping capacity, and detuning susceptibility which exhibit critical phenomena at the critical line. The origin of this critical line is elucidated by showing that it divides the parametric manifold onto two regions corresponding to different solutions of the evolution equations. The spatio-temporal behaviour of the density of trapped atoms is studied for several first topological modes.

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Figure captions

Fig. 1. The time dependence of the fractional populations $n_0(t)$ and $n_j(t)$ for $b = 0.4999$ and $\delta = 0$. Here and in the Figures 1 to 5 the dashed line corresponds to the ground-state population $n_0(t)$ and the solid line, to the excited-level population $n_j(t)$.

Fig. 2. Flattening of the fractional populations, with their oscillation period being doubled, at $b = 0.4999$ and $\delta = 0.0001$.

Fig. 3. The appearance of the upward cusps of $n_j(t)$ and of the downward cusps of $n_0(t)$ for $b = 0.4999$ and $\delta = 0.0001001$.

Fig. 4. Fractional populations versus time for $b = 0.4999$ and $\delta = 0.00011$.

Fig. 5. The dynamics of fractional populations, for varying δ and fixed $b = 0.3$, demonstrating the qualitative change of behaviour when crossing the critical line: (a) $\delta = 0.242681$; (b) $\delta = 0.242682$; (c) $\delta = 0.243000$; (d) $\delta = 0.3$.

Fig. 6. The spatio-temporal behaviour of the envelope density (18) as a function of the radial variable r , at fixed $z = 0$ for the radial dipole mode ($n = 1$, $m = 0$, $k = 0$). Here and in all following figures the characteristic parameters $g = 100$, $\nu = 10$, and $b = 0.4999$ are fixed, while the detuning δ is varied in the vicinity of the critical line $b + \delta \simeq 0.5$. In this Figure, we set $\delta = 0.00010$. Time is measured in dimensional units explained in the text: $t = 0$ (solid line); $t = 3$ (long-dashed line); $t = 15$ (short-dashed line).

Fig. 7. The same as in Fig. 6, but for the detuning $\delta = 0.00011$ and for the moments of time: $t = 0$ (solid line); $t = 10$ (long-dashed line); $t = 26$ (short-dashed line). At the time $t = 26$ the system is in the pure radial dipole state.

Fig. 8. Excitation of the basic vortex mode ($n = 0$, $m = 1$, $k = 0$). The envelope density is shown as a function of the radial variable r for $z = 0$. The detuning is $\delta = 0.00010$. The moments of time are: $t = 0$ (solid line); $t = 3$ (long-dashed line); $t = 15$ (short-dashed line).

Fig. 9. The same as in Fig. 8, but for the detuning $\delta = 0.00011$. The moments of time are: $t = 0$ (solid line); $t = 10$ (long-dashed line); $t = 26$ (short-dashed line). At $t = 26$ the system is in the pure vortex state.

Fig. 10. Excitation of the axial dipole mode ($n = 0$, $m = 0$, $k = 1$). The envelope density as a function of the axial variable z at the point $r = 0$. Here $\delta = 0.00010$. The time moments are: $t = 0$ (solid line); $t = 3$ (long-dashed line); $t = 15$ (short-dashed line).

Fig. 11. The same as in Fig. 10, but for the detuning $\delta = 0.00011$ and for the times: $t = 0$ (solid line); $t = 20$ (long-dashed line); $t = 26$ (short-dashed line). At the moment $t = 26$ the system is in the pure axial dipole state.